Horizons - International Scientific Journal • Vol. 3, No. 1, 2026 • ISSN: 2955-2303
Published by the University "St. Kliment Ohridski" – Bitola

(*) Corresponding author
Peer review method: Double-blind
Research article
DOI: https://doi.org/10.20544/hisj.2026.595
Received: 24.09.2025 • Accepted after revision: 20.12.2025 • Published: 26.02.2026

ALLPASS-BASED DUAL-NOTCH FILTER WITH CLOSELY SPACED NOTCH FREQUENCIES

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Abstract

In this paper, the method for designing a digital filter with two closely spaced notch frequencies is presented. The filter is implemented using two all-pass filters connected in parallel. To achieve an approximately linear phase, one of the all-pass filters is a pure delay. The proposed configuration, consisting of two IIR all-pass filters, can provide two notch frequencies and a flexible phase response. Unlike a filter of the same configuration with a piecewise-constant phase, where the filter order is 4 to realize two notch frequencies, the described approach for designing an approximately linear-phase filter allows increasing the filter order to reduce the approximation error.

Keywords: *Notch Filters; Parallel Connection; Approximately Linear Phase; All-Pass Filters; Equiripple Phase Error; Equiripple Magnitude Error*

INTRODUCTION

Digital multiple-notch filters are used in a variety of applications to remove or suppress multiple sinusoidal or narrow-band interference in digital signals. These applications include image processing, audio signal processing, communication, control, and biomedical engineering *etc*. A typical example is to cancel 50 Hz power line interference in the recording of electrocardiograms(Kaur and Arora 2010). In two-dimensional case notch filter eliminates a 2-D sinusoidal interference pattern superimposed on an image (Gonzales 2002); (Pei and Tseng 2010). Notch filters can be implemented as either non-recursive (Deshpande, Jain, and Kumar 2004); (Dutta Roy, Jain, and Kumar 1994) or recursive filters (Tseng 2001). The procedure described in (Dutta Roy *et al.* 1994) is for designing digital FIR notch filter for given notch frequency and 3-dB rejection bandwidth.

Usually, the IIR notch filter is preferred when a narrow notch bandwidth is required, as it involves fewer computations. Due to their significance in numerous applications, methods for their design continue to be developed.In (Kim and Nam 2008), a sharp notch filter design with a closed-form solution for filter coefficients is proposed. A modified sampling kernel is introduced and utilized along with a complementary filter concept. When designing multi-notch filters, it is important to distinguish between cases where the notch frequencies are arbitrarily distributed and those where all notch frequencies are integer multiples of a fundamental frequency.The computationally efficient method proposed by (Bruijnen *et al.* 2006) filters a harmonic series of a given fundamental frequency using multirate filter banks. An efficient approach is also proposed for implementing a large number of IIR notch filters to suppress harmonics. The multirate digital harmonic notch filter exhibits less overshoot compared to the equivalent single-rate filter.

Filters with one or multiple notch frequencies can also be implemented using all-pass filters (Nikolić, Stančić, and Cvetković 2018); (Krstić 2021). Papers (Vazkuez and Dolecek 2006) and (Zahradnik and Vlcek 2006) provide procedures for obtaining both maximally flat and equiripple magnitude characteristics.

METHODS

The procedure and results of the double notch filter design will be presented in the following sections. The notch filter will be implemented using a parallel structure, where both branches contain all-pass filters.This configuration allows the implementation of filters with arbitrary phase characteristics, including approximately constant or linear phase in the passbands. The study focuses on filters with linear phase and closely spaced notch frequencies. The procedure can be easily generalized for the design of filters with an arbitrary number of notch frequencies.

Notch Filter Configuration

The transfer function of the notch filter is given by Eq. (1) based on the configuration shown in Fig 1.

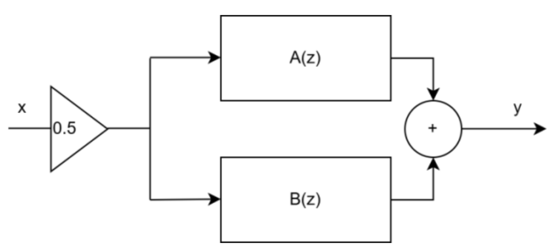

Figure 1: Approximately Linear-Phase Dual-Notch Filter Realized Using All-Pass Filters (Source: Authors' depiction)

$$H(z) = Y(z) / X(z) = 0.5(A(z) + B(z)) \tag{1}$$

In the general case, $A(z)$ and $B(z)$ are IIR all-pass filters, allowing the realization of filter with phase characteristic of arbitrary shape. On the unit circle $z = e^{j\omega}$ frequency response of the all-pass filters can be given by

$$A(e^{j\omega}) = | A(e^{j\omega}) | e^{j\varphi_A(\omega)} = e^{j\varphi_A(\omega)} \tag{2}$$

$$B(e^{j\omega}) = |B(e^{j\omega})| e^{j\varphi_B(\omega)} = e^{j\varphi_B(\omega)} \tag{3}$$

which substituted in equation (1) leads to expressions for magnitude and phase frequency response of notch filter, namely

$$| H(e^{j\omega}) | = | \cos\frac{\varphi_A(\omega) - \varphi_B(\omega)}{2} | \tag{4}$$

$$\varphi_H(\omega) = \frac{\varphi_A(\omega) + \varphi_B(\omega)}{2} \tag{5}$$

Equation (5) confirms that the notch filter phase response can be fully controlled in the process of all-pass filters design. Practically, one need to choose the same shape for the all-pass filters phases as that of the notch filter. The ideal notch filter, by definition, provides zero gain at the specified frequency and unity gain over the rest of the frequencyrange $[0, \pi]$. We recognize two passbands separated by a narrow stopband. Similarly, a dual-notch filter has three passbands and two narrow stopbands. According to equation (4),the passbands of a notch filter are located at frequencies where the phase difference between the all-pass filters satisfies the condition $\varphi_A(\omega) - \varphi_B(\omega) = 2k\pi$ in the ideal case, where k is an integer. In the dual-notch case, the phases of the all-pass filters within the three passbands exhibit mutual offsets of 0, 2π, and 4π radians, respectively. According to this information one can conclude that notch filter design turns into all-pass filters phase approximation.Based on this, it can be concluded that the design of a notch filter reduces to the problem of approximating the phase response of all-pass filters.

Two notches are realized at frequencies where the all-pass filters phase difference is exactly π and 3π radians. In a large number of practical filter applications, linear phase is of crucial importance. In such cases, the filter $A(z)$ reduces to an Mth-order delay line (a pure delay element)

$$A(z) = z^{-M} \tag{6}$$

with an ideal linear phase $\varphi_A(\omega) = -M\omega$. Due to the required phase difference of 4πradians between the two all-pass filters, the filter $B(z)$ has an order of$N$=M+4. In general, the configuration shown in Fig. 1 requires$N$=M+2K, where K is the number of notches.

Notch Filters with Constant Phase

The structure shown in Fig. 1 has the lowest complexity when the goal is to achieve the constant phase in passband (Krstić, Vasković, and Jovanović 2022). Design of multiple notch filters is presented in paper (Joshi and Dutta Roy 1999) using an all-pass filter of order $2N$, where N represents the number of notch frequencies. The design method ensures that the realized 3-dB bandwidths are lower than those specified.

The usual way to obtain a multiple notch filter is to cascade single notch filters. Such configuration may introduce the non-unity passband gains between notch frequencies. One solution is an all-pass based design using the pole-reposition technique combined with the Secant method applied to improve the performance of the optimal search algorithm (Phongsuwan, Sitthiwongwanich, Thamrongmas, and Charoenlarpnopparut 2011).

In this case, A(z)=1, and the all-pass filter B(z) is of the fourth order. This procedure yields a minimum-order multi-notch filter. Since the phase of A(z) is zero at all frequencies, notch frequencies are obtained where the phase of B(z) exactly equals $-\pi$ and -3π. This configuration does not allow for reducing phase and magnitude errors by increasing the filter order, as doing so would introduce additional notches. This issue does not arise when implementing anall-pass filterwith linear-phase response.

All-Pass Filter

The transfer function of IIR all-pass filter $B(z)$ is given by

$$B(z) = \frac{b_N + b_{N-1}z^{-1} + ... + b_1 z^{-N+1} + z^{-N}}{1 + b_1 z^{-1} + b_2 z^{-2} + ... + b_N z^{-N}} \tag{7}$$

The corresponding phase response is given by

$$\varphi_B(\omega) = -N\omega + 2\arctan\frac{b_1 \sin(\omega) + ... + b_N \sin(N\omega)}{1 + b_1 \cos(\omega) + ... + b_N \cos(N\omega)} \tag{8}$$

Given that the design of the notch filter is carried out using the phase approximation of the all-pass filter, the effect of zero–pole pairs on the phase is presented below.

The phase response of the all-pass filter, along with its corresponding zeros and poles, is shown in Fig. 2. The poles of the all-pass filter are chosen to be located at $\rho = 0.98\exp(\pm j\pi/7)$ and $\rho = 0.75\exp(\pm j3\pi/4)$. As is well known, the total phase variation from zero frequency to the Nyquist frequency directly depends on the filter order N and equals $N\pi$, which, in the case of the proposed dual-notch filter,is 4π.

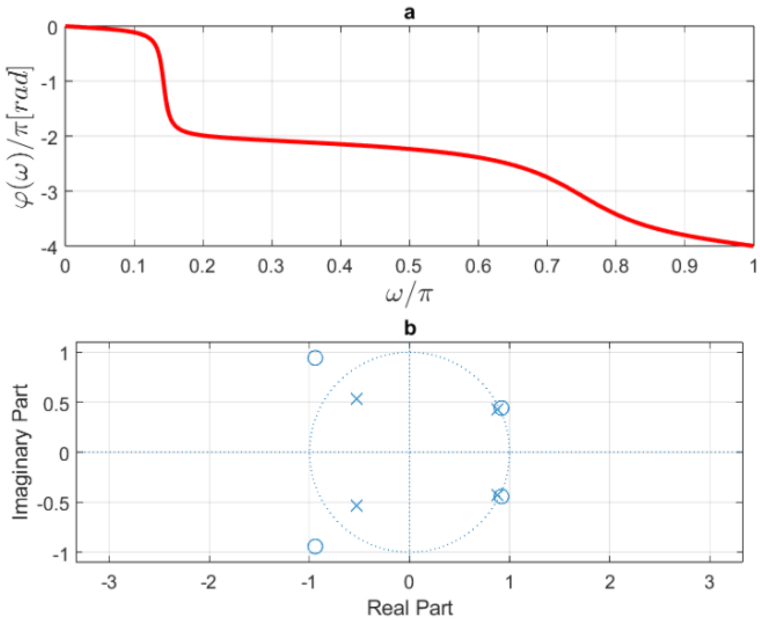

Figure 2: (a) Phase response of the fourth-order all-pass filter and (b) zeros and poles of the transfer function (Source: Authors' depiction)

The most significant phase change occurs in the vicinity of the pole frequency, as shown in Fig. 2. For a conjugate complex pole pair, the total phase shift over the interval [0, π] is 2π radians. This implies that a second-order section can provide only a single notch at the frequency where $\varphi_B = -\pi$ for the case A(z)=1. It should be recalled that this work focuses on the case of equiripple phase error, which also leads to an equiripple magnitude error while aiming for a linear phase response. The least hardware-demanding configuration is obtained by choosing one all-pass filter as a pure delay element with ideal linear phase. By implementing a 2π phase jump in the second all-pass filter, the transition between two passbands is achieved in the magnitude response, with zero gain realized exactly at the frequency where the phase difference equals π radians.

In the case of equiripple error, the error function reaches a maximum just before the abrupt phase change and a minimum immediately after. This means that if a 2π phase shift needs to be realized, the total phase variation in a narrow region exceeds 2π, which cannot be achieved with a second-order section. This limitation does not exist in the realization of a lowpass filter, where a π radian phase shift is sufficient to transition from the passband to the stopband. The problem is elegantly solved by choosing a double conjugate complex pole that exhibits a total phase variation of 4π over the interval [0, π], allowing the narrowband phase jump slightly exceeding 2π (the approximation error is small compared to 2π). Naturally, we assume that, for the realization of a single notch, the delay line must be second-order.

For poles located closer to the unit circle, the phase transition becomes steeper compared to poles further from the unit circle, resulting in a narrower band-reject region. In the design of the dual-notch filter, it can be intuitively concluded that it is sufficient to take the previously described configuration twice, *i.e.*, two double complex conjugate poles combined with a fourth-order delay line. Two notches are obtained at the frequencies where the phase of the all-pass filter deviates by π and 3π radians from the ideal linear phase. The difference in order between the two all-pass filters must be exactly four to prevent the creation of more than two notch frequencies.

Applied to a filter with two closely spaced notch frequencies, this approach reduces to using only a single complex pole, but with quadruple multiplicity, as demonstrated in (Nikolić, *et al.* 2018).

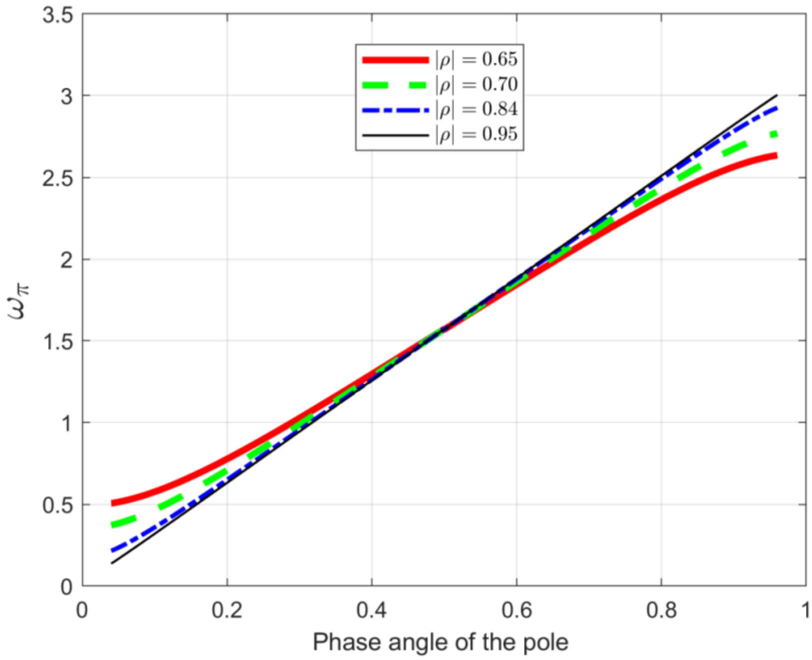

Figure3: Dependence of the frequency ω_π , at which the phase of the second-order all-pass filter reaches $-\pi$, on the pole phase angle (Source: Authors' depiction)

Regardless of the pole magnitude, the phase change is greatest in the vicinity of the pole frequency, which is particularly noticeable for magnitudes above 0.8. It should be noted that the frequency at which the phase of the second-order all-pass filter reaches −π does not coincide exactly with the pole frequency; this relationship is illustrated in Fig. 3. The illustrated dependence facilitates the selection of initial pole value for the all-pass filter corresponding to a specified notch frequency.

The total area under the group delay curve over the interval [0,π] for an all-pass filter of order N is equal to Nπ.A linear phase corresponds to a constant group delay, which sometimes makes it simpler to obtain an initial solution for the all-pass filter by analyzing the group delay. The ideal notch filter obtained from the parallel structure shown in Fig. 1 would have a constant group delay in both passbands. It is well known that the approximation error decreases with the increasing order of the transfer function polynomials, *i.e.*, with the increasing number of poles over the considered frequency range. Therefore, to achieve approximately identical approximation errors in both passbands, the ratio of the number of poles per region should roughly correspond to the ratio of the bandwidths of these regions.

Linear Phase Initial Solution

Considering the phase characteristics of all-pass filters, itcan be concluded that a phase jump of 4π requires the use of atriple complex pole in design of filter with two close notches. It is easy to arrive at an initial solution that provides a number of extrema equal to the number of unknowns (transfer function coefficients). We consider equiripple phase error case. If all poles are simple and uniformly spaced on a circle of radius less than unity (to satisfy the stability condition), the phase error (deviation from linear phase) exhibits equiripple behavior, with the amplitude of the maximum deviation being directly determined by the radius of the pole circle. This relationship is shown in Fig. 4 for various values of the all-pass filter order N .

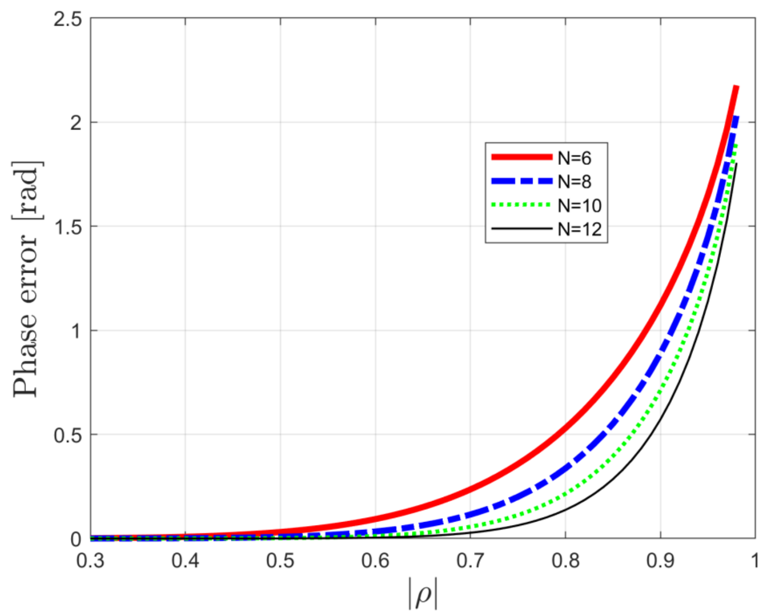

Figure 4:Dependence of the Maximum Phase Error on the Pole Modulus for Various All-Pass Filter Orders (Source: Authors' depiction)

This is the first step in design of the low-pass, high-pass, *etc*. filters (utilizing the configuration given in Fig. 1), where a phase jump of π radians transitions into the next band (either passband or stopband). Notch filters require a slightly different approach, as we want a narrow stopband with a rapid return to the passband.

For the realization of a single notch, a total phase shift of 2π radians is required, resulting in a notch at the frequency where the phase difference between the two all-pass filters equals π radians. The issue is that a complex-conjugate pole pair, as mentioned earlier *(i.e*., a second-order all-pass section), provides a total phase shift of 2π radians only over the entire frequency range from 0 to the Nyquist frequency. Therefore, a double pole is required, meaning that a fourth-order section must be employed.The initial solution is obtained by positioning the poles equidistantly, where all simple poles are spaced $2\pi/N$ apart, while the triple pole is positioned at a distance of $4\pi/N$ from its neighboring poles. m_1 and m_2 are the number of phase error extrema in passbands located left and right from two notches, respectively. For the case N=12, with

$m_1 = 3$ and $m_2 = 5$, a single pole is placed at the lowest frequencies with a phase angle of π/N. The phase jump required for the realization of two notches is provided by the pole whose initial phase angle equals 5π/N. In the second passband, two simple poles are located at the frequencies 9π/N and 11π/N. The phase and corresponding phase error of the all-pass filter, for double notch filter design, obtained by triple pole are presented in Fig 5.

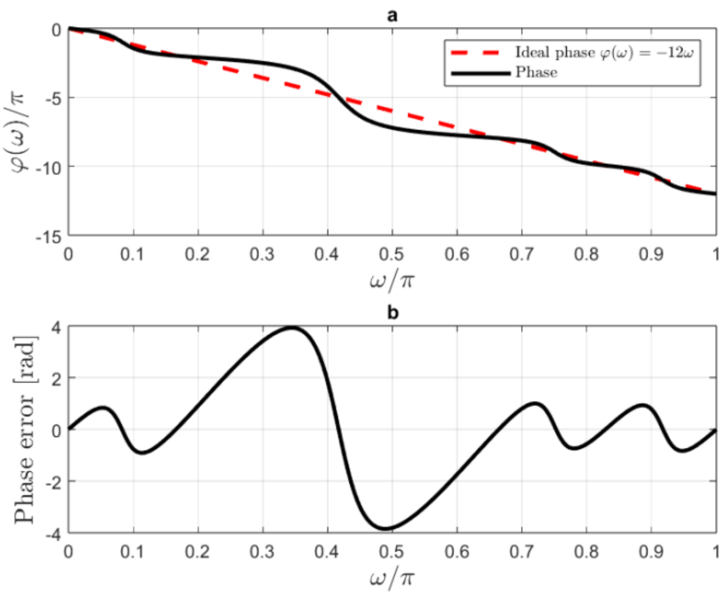

Figure 5: (a) The phase of initial solution for all-pass filter of order *N*=12 and (b) corresponding phase error (Source: Authors' depiction)

The poles of the initial solution shown in Fig. 5 are located on a circle with a radius of 0.93. It is advisable to choose the pole circle radius within the range $0.85 < r < 0.98$, especially for higher-order all-pass filters, where the phase error curve contains a large number of extrema. Although all poles are equally distant from the origin, in the case of multiple poles the corresponding error extrema attain higher values compared to the others. To achieve a more uniform distribution of extrema, a slight modification of the initial solution can be made by moving the multiple pole closer to the origin while keeping its phase angle unchanged.

The number of extrema of the phase error is equal to the number of unknown coefficients of the all-pass filter. The procedure allows the permissible phase error εto be determined ($\varepsilon = \varepsilon_1 = \varepsilon_2$) for a desired attenuation a_{dB} in the passbands, according to

$$\varepsilon = 2\arccos(10^{-a_{dB}/20}) \tag{9}$$

which results in equal phase deviations in both passbands, although the locations of the two notch frequencies are not controlled. This provides a rough estimate of the phase error for a given filter order, which can be reduced by increasing the filter order. By adding two additional equations related to the notch frequencies into the system of equations, it becomes possible to optimize the phase error in both passbands and obtain the optimal solution for the given filter order.

If the dual-notch filter does not satisfy the specified phase-error requirements, the problem can be solved by selecting a filter with a higher number of poles, *i.e.*, more extrema in the problematic passband. Alternatively, one may first attempt to relocate the extrema from one passband to the other while keeping the same all-pass filter order; if this approach fails, it becomes necessary to increase the complexity of the all-pass filter.

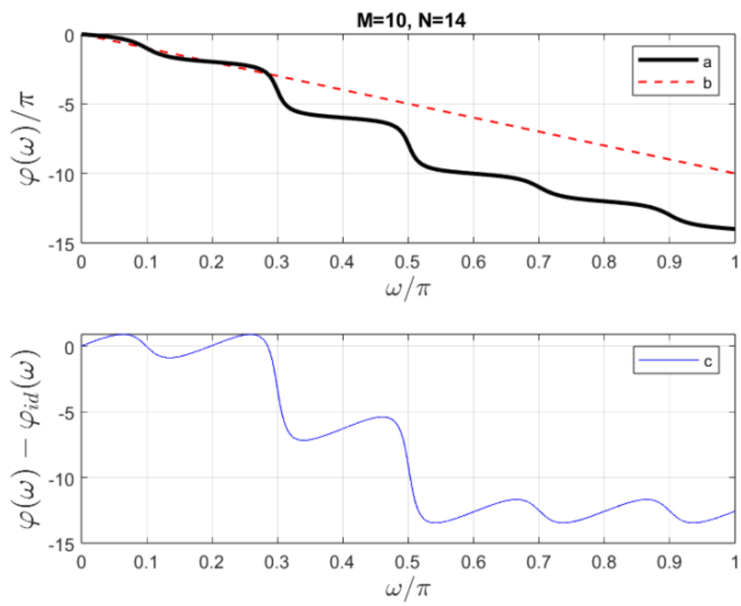

Figure 6: (a) The phase of all-pass filter, (b) phase of delay line, and (c) corresponding phase error (Source: Authors' depiction)

It is possible to obtain two notches using two double poles. This solution is shown in Fig.6. This filter also has 3 *i.e*. 5 extrema in lower and upper passband but triple pole solution need delay line of order 8 while filter with two double poles apply delay line of order 10. From Fig. 6, one minimum and one maximum can be observed, located between the two notch frequencies.

Designing a multiple notch filter reduces to designing an all-pass filter by approximating its phase response, which in turn approximates an ideal linear phase in the min-max sense. Unknown poles (moduli and phase angles) or second-order section coefficients are determined by solving the following system of equations.

$$\begin{aligned} \varphi_B(\omega_i) + M\omega_i &= (-1)^{i+1}\varepsilon_1, i = 1,2,\dots,m_1 \\ \varphi_B(\omega_i) + M\omega_i + 4\pi &= (-1)^{i}\varepsilon_2, i = 1,2,\dots,m_2 \\ \varphi_B(\omega_i) + M\omega_i + (2i-1)\pi &= 0, i = 1,2 \end{aligned} \tag{9}$$

The procedure for solving the system of equations (9) is described in detail in (Nikolić*et al.* 2018). The last two equations in (9) refer to notch frequencies where phase error must be zero.

RESULTS

The proposed method has been successfully used for designing filters of order up to 50. A higher number of poles, *i.e*. more extrema in the passband, ensures lower phase and amplitude errors, which are directly related and both exhibit an equiripple behavior in all passbands.For given values of two notch frequencies, an appropriate selection of the number of extrema in both bands can achieve approximation errors of similar magnitudes. The presented results correspond to the case where$N$=12, M=8, m_1=3, and m_2=5.

The phase characteristics of the two obtained filters are shown in Fig. 7. The filter labeled as (a) has its lower notch at 0.38π, while the filter labeled as (b) has it at 0.36π. In both cases, the upper notch is positioned at 0.4π. Their magnitude characteristics are shown in Fig. 8, including the magnitude error in the passbands. It can be observed that small changes in the notch frequency, i.e., their differences, cause significantvariationsin the obtained attenuation in the passbands. For $\omega_{n1} = 0.36\pi$, the attenuation is 0.0025 dB, while for $\omega_{n1} = 0.38\pi$, its value is 0.35 dB.

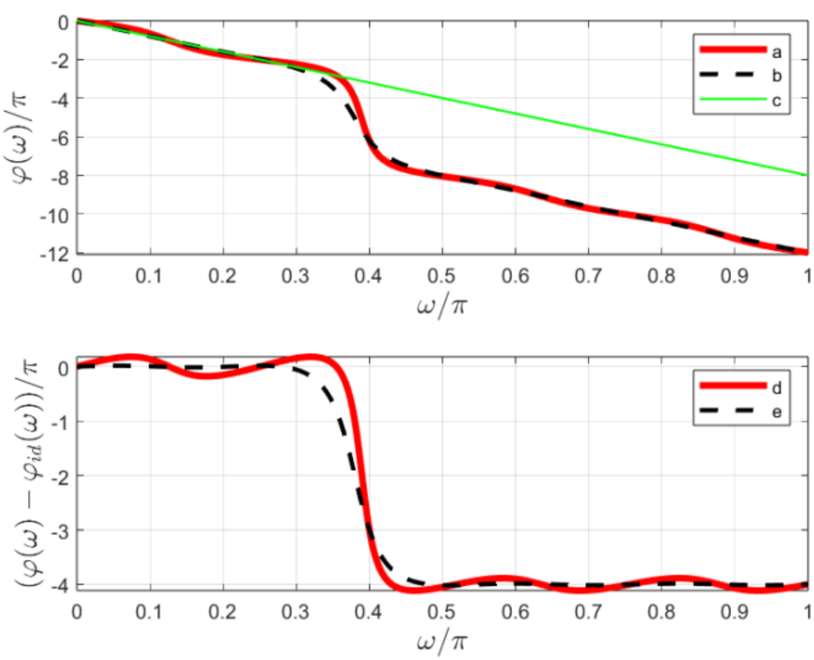

Figure7: (a,b) Phase response of all-pass filters, (d,e) corresponding errors, and (c) delay line phase (Authors' depiction)

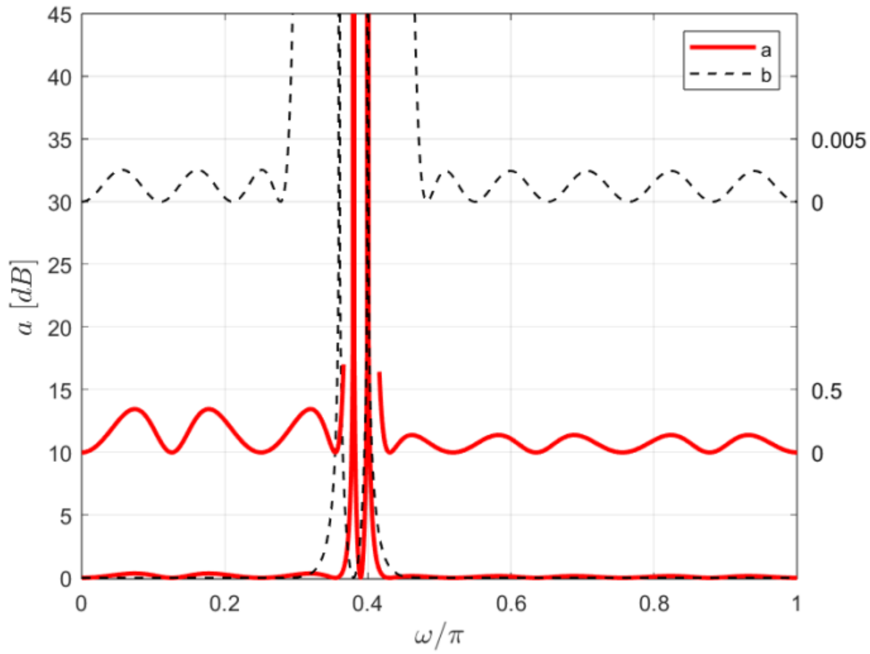

Figure 8: Attenuation of dual notch filters with (a) $\omega_{n1} = 0.38\pi$ and (b) $\omega_{n1} = 0.36\pi$
(Source: Authors' depiction)

To compare this filter with a non-recursive solution, an FIR filter of order N=600 was designed with notch frequencies at 0.36π and 0.4π, using the frequency sampling method. The frequency and corresponding gain vectors, each of length 11, used as input arguments for FIR filter design, were selected based on Fig. 8. Four frequencies corresponding to 3 dB attenuation were read, in addition to the boundary frequencies where the gain should be equal to one (frequencies where the gain in Fig. 8 exceeds 0.999) or zero (the notch frequencies).The corresponding magnitude response is shown in Fig. 9. Despite the linear phase, which FIR filters can easily achieve through coefficient symmetry, Fig. 9 shows that the amplitude response is still unsatisfactory, even considering the high filter complexity. All filter characteristics in this work are shown with the same resolution, *i.e.*, 1000 points were used over the interval [0,π].

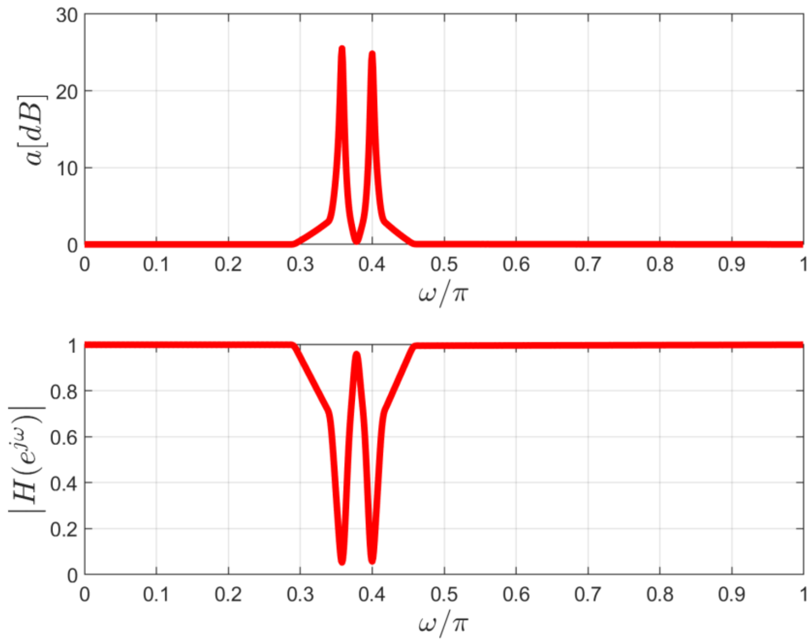

Figure 9: Attenuation of dual notch FIR filter of order $N = 600$ with $\omega_{n1} = 0.36\pi$ and $\omega_{n2} = 0.40\pi$ (Source: Authors' depiction)

Fig. 10 depicts how the spacing between notch frequencies varies with maximum phase error in the upper passband for choosen phase error in lower passband.The maximum phase error in both passbands is approximately 10 degrees when the notch frequencies differ by 0.03.

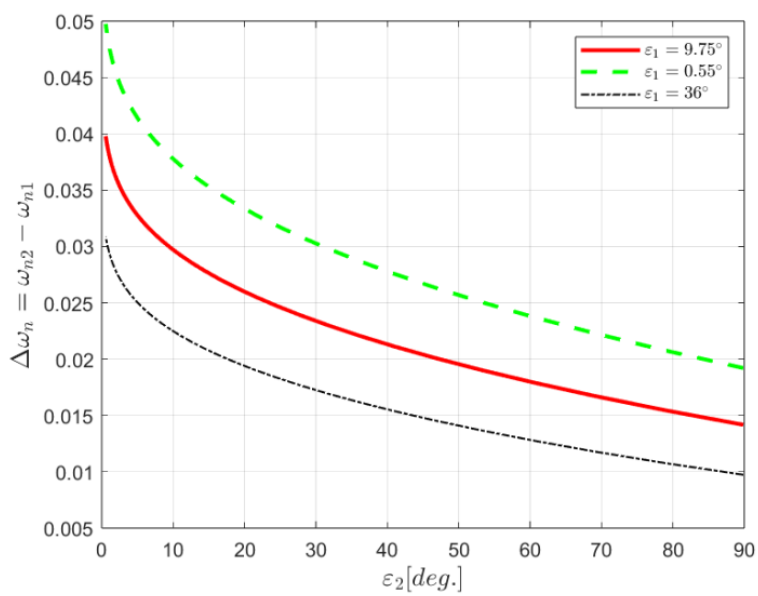

Figure 10: Dependence of notch frequency spacing on the maximum phase error ε_2, shown for various values of ε_1 (Source: Authors' depiction)

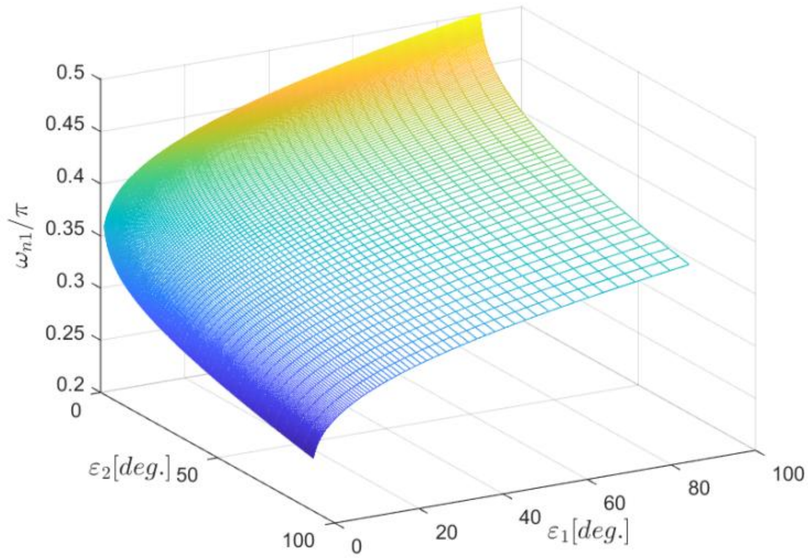

Figure 11: Notch frequency ω_{n1} as a function of the phase errors ε_1 and ε_2 in the passbands (Source: Authors' depiction)

The dependence of notch frequency ω_{n1} and the difference between closely spaced notch frequencies on the phase error in the passbandsare shown in Figs. 11 and 12, respectively.

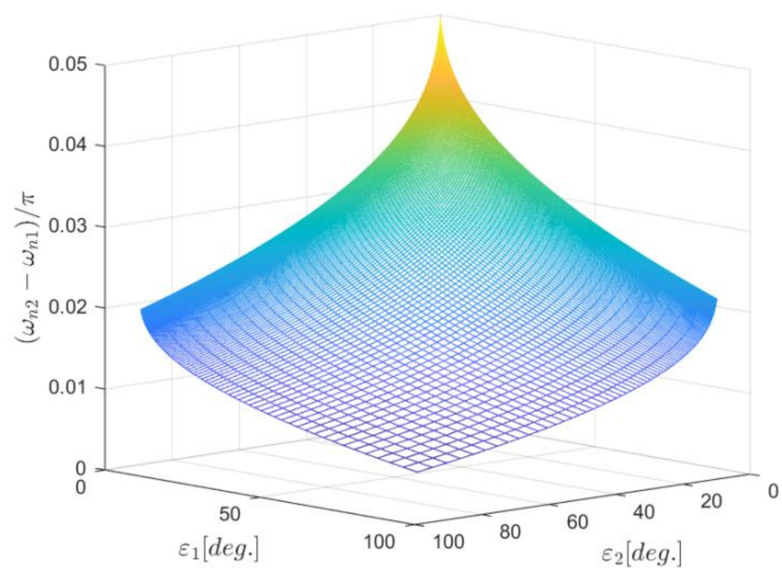

Figure 12: Notch frequency difference $\Delta\omega_n$ as a function of the phase errors ε_1 and ε_2 in the passbands(Source: Authors' depiction)

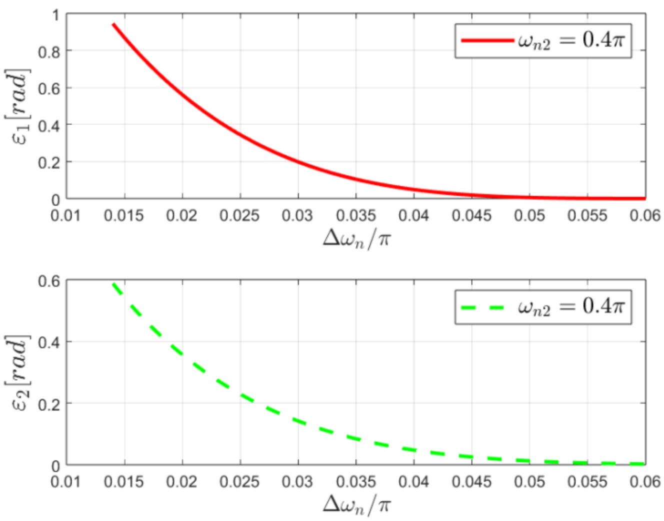

Figure 13: Dependence of the maximum phase error in passbands ε_1 and ε_2 on the distance between notch frequencies $\Delta\omega_n$ for $\omega_{n2} = 0.4\pi$ (Source: Authors' depiction)

For $\Delta\omega_n < 0.0347\pi$, the phase errors ε_1 and ε_2 are less than 5 degrees. Figure 13 shows the maximum phase errors ε_1 and ε_2 as functions of the notch frequency spacing, for $\omega_{n2} = 0.4\pi$.

Dual-notch filters with identical attenuation in both passbands, ranging from 0.01 dB to 1 dB, and with parameters m_1 =3, and m_2 =5, were implemented using the described procedure for the case of a triple pole and for the case of a quadruple pole as in (Nikolić *et al.*). The obtained results are presented in Fig. 14.

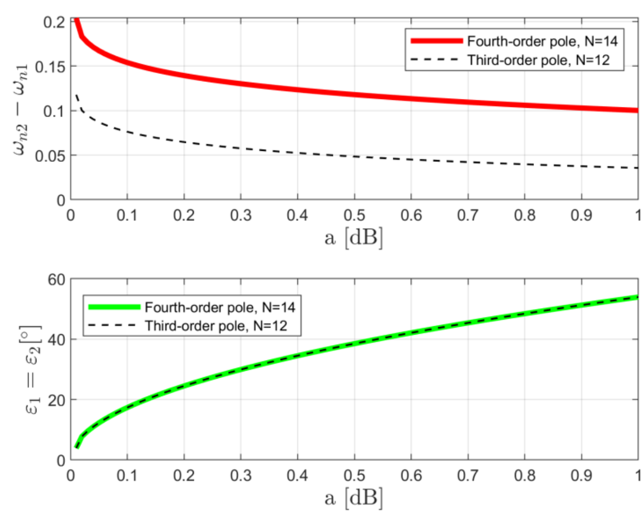

Figure 14: Separation between notches as a function of passband attenuation for filters with triple and quadruple poles, with m_1 =3 and m_2 =5(Source: Authors' depiction)

Regardless of the difference in complexity between the two notch filters, both have the same number of extrema on the phase error curve, resulting in nearly identical values for the maximum approximation error. However, the proposed filter exhibits notch frequencies that are approximately 0.022πcloser to each other.The lower plot corresponds to the relation given by equation (9). When the permissible phase error is below 10 degrees, the designed filter shows extremely low attenuation within the passbands.

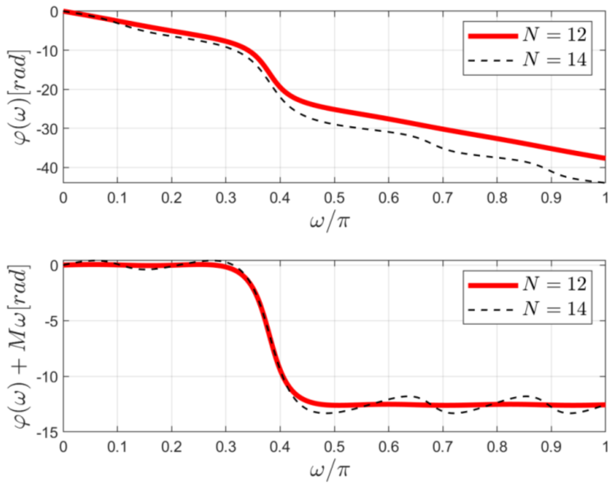

Figure 15: Phases and phase differences of the all-pass filters of order N and the corresponding delay line of order M for m_1 =3 and m_2 =5(Source: Authors' depiction)

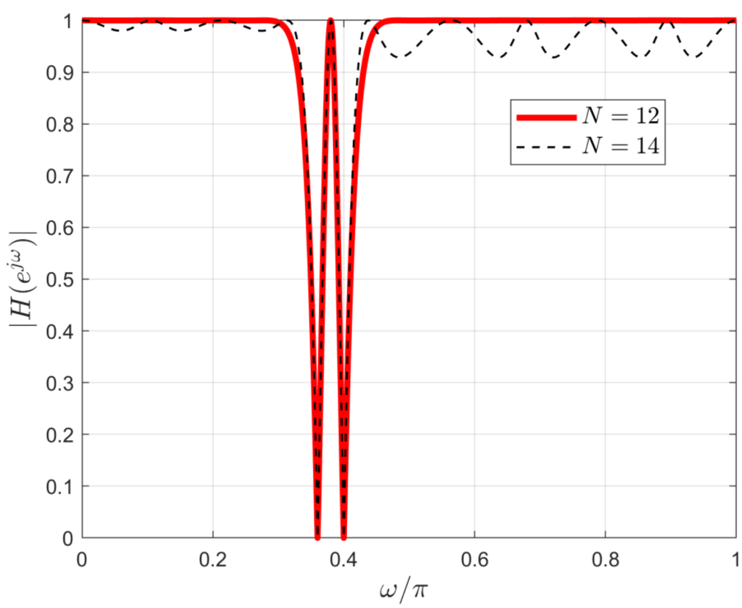

Figure 16: Magnitude of notch filters obtained by all-pass filters of order N and the corresponding delay lines of order M for m_1 =3 and m_2 =5(Source: Authors' depiction)

Figures 15 and 16 show the characteristics of filters with notches at 0.36π and 0.40π. Despite the lower order of the filter using a triple pole and the identical number of extrema in the amplitude and phase error curves, the filter obtained by the proposed procedure exhibits significantly lower attenuation in the passbands.

A filter with two notch frequencies can be realized by cascading two filters, each independently producing one of the notch frequencies. A single notch frequency can be obtained, for example, using a parallel connection of a fourth-order all-pass filter with a double complex pole and a second-order delay line. By cascading two such filters, a dual-notch filter is obtained. The result is shown in Fig. 17 for $\omega_{n1} = 0.36\pi$ and $\omega_{n2} = 0.40\pi$. As can be observed from Fig. 17, the resulting filter exhibits a minimum attenuation of about 25 dB between the two notch frequencies, where in the ideal case a passband should appear. When the proposed design procedure is applied (requiring a sixth-order all-pass filter with a triple pole) the resulting filter, despite having the minimum possible order for realizing two notches, achieves attenuation reduced to 0 dB between the notches. This occurs because the phase of the all-pass filter changes by 2π radians relative to the delay line phase between the two notch frequencies, resulting in a phase difference of 2π radians at some frequency (phase of the all-pass filter at first notch frequency is already shifted by π compared to delay line phase), which, according to equation (4), produces unity gain. The point with unity gain between two notches is observable in Fig. 16. Such points exist between every two notches in case of multiple-notch filter if it is realized by parallel all-pass structure.

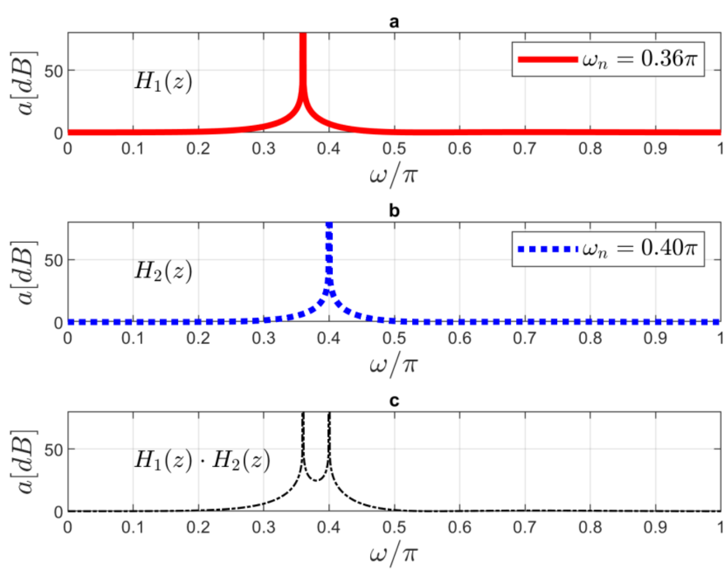

Figure 17: Attenuation of notch filters based on a fourth-order all-pass filter: (a), (b) single-notch filters; (c) their cascade connection (Source: Authors' depiction)

CONCLUSION

A method for design of IIR filter with two close notch frequencies is presented in this paper. Filter has parallel structure with delay line and all-pass filter positioned in two branches. The resulting phase response is near-linear. Triple complex pole is simple solution to obtain system of equations of the same order as number of unknown poles. In the case where two notches are obtained using two double poles, one or two phase error extrema may be missing for closely spaced notches, making it impossible to form a sufficient number of equations to determine the transfer function coefficients. The proposed method overcomes this problem.

Unlike the case of constant phase, this solution allows approximation errors to be reduced by selecting higher-order all-pass filters with delay line of corresponding order. The described procedure refers to solving the problem of realizing two closely spaced notch frequencies but can be easily modified to design a filter with an arbitrary number of notch frequencies.

Compared to the solution that uses two double poles or one quadruple pole, the described procedure provides significantly lower phase error (and therefore attenuation closer to zero in the corresponding regions) while saving hardware, since both the IIR all-pass filter and the delay line have transfer functions of two orders lower.

CRediT AUTHOR STATEMENT

Petar Stančić: Investigation, Writing – original draft, Review & editing. **Goran Stančić:** Conceptualization, Investigation, Writing – original draft. **Ivana Đorđević:** Review & editing. **Stevica Cvetković:** Supervision, Methodology.

All authors have read and agreed to the published version of the article.

COMPLIANCE WITH ETHICAL STANDARDS

Acknowledgments:
Not applicable.

Funding:
This work was supported by the Ministry of Science, Technological Development and Innovation of the Republic of Serbia under Grant No. 451-03-34/2026-03/200102.

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The authors acknowledge the use of AI-based language tools (ChatGPT) to assist in improving the English language and readability of the manuscript. These tools were used solely for linguistic refinement and were not involved in the scientific content, analysis, or conclusions. The authors take full responsibility for the integrity, originality, and accuracy of the work.

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